Lyapunov-Malkin Theorem and Stabilization of the Unicycle with Rider

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Abstract

In this paper we discuss stabilization of a nonholonomic system consisting of a unicycle with rider. We show in particular that one can achieve stability of slow steady vertical motions by imposing a feedback control force on the rider's limb.

1 Introduction

In this paper we study the stabilization problem for a model of a rider on a unicycle using some of the ideas discussed in Zenkov, Bloch and Marsden [1998]. In that paper we analyzed various techniques for studying the stability of motion of nonholonomic mechanical systems. In particular we considered energy-based methods as well as use of the so-called Lyapunov-Malkin theorem (see below). In both cases we used the special structure of nonholonomic mechanical systems with symmetry, where we can divide up the system variables into internal (or shape) variables and momentum variables corresponding to symmetry directions. However, unlike holonomic systems, symmetries do not lead via Noether's theorem to conservation laws, and in general momenta corresponding to symmetries obey dynamic momentum equations (see Bloch, Krishnaprasad, Marsden and Murray [1996]). In some cases we were able to show that stability of motion could nonetheless be analyzed using a generalization of energy-momentum methods (see e.g. Marsden [1992]). In other cases we used a combination of spectral and nonlinear analysis.

Here we model the rider on a unicycle in this paper by a double pendulum on a wheel, the two pendula representing the body and the limb of the rider. This leads to a complicated but tractable equations. We then apply linear control to the pendulum representing the limb of the rider, but conclude nonlinear stability of the overall system using the Lyapunov-Malkin theorem. This theorem, which enables us to conclude overall nonlinear stability using partial spectral information about the system, turns out to be particularly useful for the analysis of nonholonomic systems (see Karapetyan [1981], Markeev [1992], Zenkov, Bloch and Marsden [1998]). In particular here we apply this technique to achieve stabilization of slow vertical steady state motions of a homogeneous disk on a horizontal plane with a double pendulum attached. (Fast motions may also be stabilized and are in fact easier to handle because of the stabilizing effect of the wheel velocity.)

While the analysis here is quite nontrivial in itself we intend to extend it both to more complex nonholonomic/robotic systems and to more complicated nonlinear control techniques, for example the matching control technique discussed in Bloch, Leonard and Marsden [1997, 1998].

2 Modeling the unicycle with rider

We derive here the dynamics of a homogeneous disk on a horizontal plane with a double pendulum attached. The upper pendulum is free to move in the plane orthogonal to the disk while the lower pendulum stays "vertical" in the disk's plane. We view this as a simple model of a rider on a unicycle. See Figure 2.1 for details. We accept the following notations:

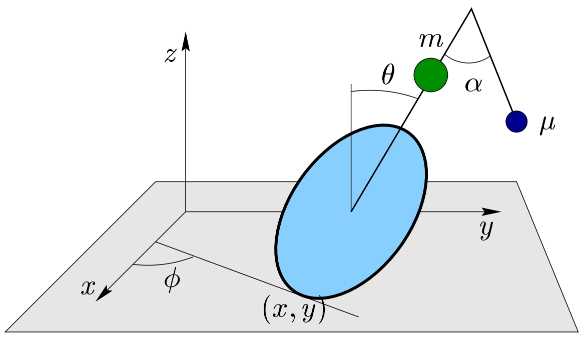

Figure 2.1: The disk on the horizontal plane.

$$
\begin{aligned}
M &= \text{the mass of the disk,}\\
R &= \text{the radius of the disk,}\\
A, B &= \text{the principal moments of inertia of the disk,}\\
(x, y) &= \text{coordinates of the contact point,}\\
\theta &= \text{the angle between the disk and the vertical axis,}\\
\phi &= \text{the heading angle of the disk,}\\
\psi &= \text{the self-rotation angle of the disk,}\\
\alpha &= \text{the angle between the pendula,}\\
m &= \text{the principal pendulum bob mass,}\\
r &= \text{the principal pendulum length,}\\
l &= \text{the distance from the center of the disk to the bob,}\\
\mu &= \text{the arm mass,}\\
\rho &= \text{the arm length.}
\end{aligned}
$$

The Lagrangian of this system is

$$L = K_{\text{disk}} + \frac{m}{2} v_m^2 + \frac{\mu}{2} v_\mu^2 - U. \tag{2.1}$$

In the above formula,

$$\begin{aligned}
K_{\text{disk}} &= \frac{M}{2} \Big[\dot{x}^2 + \dot{y}^2 - 2R\dot{x}\dot{\phi} \sin\theta \cos\phi - 2R\dot{y}\dot{\phi} \sin\theta \sin\phi \\
&\quad - 2R\dot{x}\dot{\theta} \cos\theta \sin\phi + 2R\dot{y}\dot{\theta} \cos\theta \cos\phi + R^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta \Big] \\
&\quad + \frac{1}{2} \Big[A(\dot{\theta}^2 + \dot{\phi}^2 \cos^2\theta) + B(\dot{\phi} \sin\theta + \dot{\psi})^2 \Big], \\
v_m^2 &= \dot{x}^2 + \dot{y}^2 + (R+l)^2 \sin^2\theta \, \dot{\theta}^2 \\
&\quad - 2(R+l)\dot{x} \Big[\cos\theta \sin\phi \, \dot{\theta} + \sin\theta \cos\phi \, \dot{\phi} \Big] \\
&\quad + 2(R+l)\dot{y} \Big[\cos\theta \cos\phi \, \dot{\theta} - \sin\theta \sin\phi \, \dot{\phi} \Big] \\
&\quad + (R+l)^2 \Big[\cos\theta \sin\phi \, \dot{\theta} + \sin\theta \cos\phi \, \dot{\phi} \Big]^2 \\
&\quad + (R+l)^2 \Big[\cos\theta \cos\phi \, \dot{\theta} - \sin\theta \sin\phi \, \dot{\phi} \Big]^2, \\
v_\mu^2 &= \dot{x}^2 + \dot{y}^2 \\
&\quad - 2\dot{x} \Big\{ \Big[(R+r) \cos\theta \, \dot{\theta} + \rho \cos(\alpha - \theta)(\dot{\alpha} - \dot{\theta}) \Big] \sin\phi \\
&\qquad + \Big[(R+r) \sin\theta + \rho \sin(\alpha - \theta) \Big] \cos\phi \, \dot{\phi} \Big\} \\
&\quad + 2\dot{y} \Big\{ \Big[(R+r) \cos\theta \, \dot{\theta} + \rho \cos(\alpha - \theta)(\dot{\alpha} - \dot{\theta}) \Big] \cos\phi \\
&\qquad - \Big[(R+r) \sin\theta + \rho \sin(\alpha - \theta) \Big] \sin\phi \, \dot{\phi} \Big\} \\
&\quad + \Big[(R+r) \cos\theta \, \dot{\theta} + \rho \cos(\alpha - \theta)(\dot{\alpha} - \dot{\theta}) \Big]^2 \\
&\quad + \Big[(R+r) \sin\theta \, \dot{\theta} - \rho \sin(\alpha - \theta)(\dot{\alpha} - \dot{\theta}) \Big]^2 \\
&\quad + \Big[(R+r) \sin\theta + \rho \sin(\alpha - \theta) \Big]^2 \dot{\phi}^2, \\
U &= MgR \cos\theta + mgl \cos\theta + \mu g \left[(R+r) \cos\theta - \rho \cos(\alpha - \theta) \right].
\end{aligned}$$

The constraints are

$$\dot{x} = -\dot{\psi} R \cos\phi, \qquad \dot{y} = -\dot{\psi} R \sin\phi. \tag{2.2}$$

The equations of motion are

$$
\begin{aligned}
&\frac{d}{dt}\frac{\partial L_c}{\partial \dot\theta} = \frac{\partial L_c}{\partial \theta},\\
&\frac{d}{dt}\frac{\partial L_c}{\partial \dot\alpha} = \frac{\partial L_c}{\partial \alpha},\\
&\frac{d}{dt}\frac{\partial L_c}{\partial \dot\phi} = \mathcal{A}\cos\theta\,\dot\theta\dot\psi + \mathcal{B}\cos(\alpha-\theta)(\dot\alpha-\dot\theta)\dot\psi,\\
&\frac{d}{dt}\frac{\partial L_c}{\partial \dot\psi} = -\mathcal{A}\cos\theta\,\dot\theta\dot\phi - \mathcal{B}\cos(\alpha-\theta)(\dot\alpha-\dot\theta)\dot\phi.
\end{aligned}
$$

In the above equations,

$$
\begin{aligned}
L_c &= L|_{\dot x = -\dot\psi R\cos\phi,\ \dot y = -\dot\psi R\sin\phi},\\
\mathcal{A} &= MR^2 + mR(R+l) + \mu R(R+r),\\
\mathcal{B} &= \mu R\rho.
\end{aligned}
$$

Next, introduce the nonholonomic momentum and the constrained Routhian by

$$
p_1 = \frac{\partial L_c}{\partial \dot\phi}, \qquad p_2 = \frac{\partial L_c}{\partial \dot\psi}
$$

and

$$
\mathcal{R} = \frac{1}{2}\left(g_{11}\dot\theta^2 + 2g_{12}\dot\theta\dot\alpha + g_{22}\dot\alpha^2\right) - \frac{1}{2}I^{ab}p_ap_b - U(\theta,\alpha), \tag{2.3}
$$

respectively. Here

$$
\begin{aligned}
g_{11} &= MR^2 + m(R+l)^2 + \mu\left[(R+r)^2 - 2(R+r)\rho\cos\alpha + \rho^2\right] + A,\\
g_{12} &= \mu\left[(R+r)\rho\cos\alpha - \rho^2\right],\\
g_{22} &= \mu\rho^2,\\
I_{11} &= MR^2\sin^2\theta + m(R+l)^2\sin^2\theta\\
&\quad + \mu\left[(R+r)\sin\theta + \rho\sin(\alpha-\theta)\right]^2 + A\cos^2\theta + b\sin^2\theta,\\
I_{12} &= MR^2\sin\theta + mR(R+l)\sin\theta\\
&\quad + \mu R\left[(R+r)\sin\theta + \rho\sin(\alpha-\theta)\right] + B\sin\theta,\\
I_{22} &= MR^2 + mR^2 + \mu R^2 + B.
\end{aligned}
$$

See Zenkov, Bloch, and Marsden [1998] for details on nonholonomic momenta and the Routhian.

The equations of motion become

$$\frac{d}{dt}\frac{\partial \mathcal{R}}{\partial \dot{\theta}} = \nabla_\theta \mathcal{R}, \tag{2.4}$$

$$\frac{d}{dt}\frac{\partial \mathcal{R}}{\partial \dot{\alpha}} = \nabla_\alpha \mathcal{R}, \tag{2.5}$$

$$\frac{dp_1}{dt} = [I^{21}p_1 + I^{22}p_2][\mathcal{A}\cos\theta\,\dot{\theta} + \mathcal{B}\cos(\alpha-\theta)(\dot{\alpha}-\dot{\theta})], \tag{2.6}$$

$$\frac{dp_2}{dt} = -[I^{11}p_1 + I^{12}p_2][\mathcal{A}\cos\theta\,\dot{\theta} + \mathcal{B}\cos(\alpha-\theta)(\dot{\alpha}-\dot{\theta})]. \tag{2.7}$$

The first two equations here describe the motion of the double pendulum, while the second two equations model the (coupled) wheel dynamics. The covariant derivatives in equations (2.4) and (2.5) are defined by

$$\begin{aligned}\nabla_\theta &= \frac{\partial}{\partial\theta} + \Big[\mathcal{A}\cos\theta - \mathcal{B}\cos(\alpha-\theta)\Big] \\ &\qquad\qquad \Big[\big(I^{21}p_1 + I^{22}p_2\big)\frac{\partial}{\partial p_1} - \big(I^{11}p_1 + I^{12}p_2\big)\frac{\partial}{\partial p_2}\Big], \\ \nabla_\alpha &= \frac{\partial}{\partial\alpha} + \mathcal{B}\cos(\alpha-\theta)\Big[\big(I^{21}p_1 + I^{22}p_2\big)\frac{\partial}{\partial p_1} - \big(I^{11}p_1 + I^{12}p_2\big)\frac{\partial}{\partial p_2}\Big].\end{aligned}$$

3 Feedback stabilization

In this section we describe a feedback law that stabilizes slow vertical steady state motions of the unicycle with rider. We remark that fast steady state motions of the unicycle without rider do not require stabilization (Zenkov, Bloch, and Marsden [1998]). It is this fact that makes fast motions of the unicycle with rider easier to stabilize than slow motions.

We introduce a single control into the upper pendulum. One can think of this as a controlled limb of the rider. (Of course one can introduce a forward motion or steering control for the unicycle, but this is not key to the stability analysis here. Such controls for the wheel are discussed for example in Bloch, Reyhanoglu and McClamroch [1992], Bloch, Krishnaprasad, Murray and Marsden [1996] and references therein.)

Our stability analysis is based on the following theorem:

Theorem 3.1 (Lyapunov-Malkin) *Consider the system of differential equations*

$$\dot{x} = Ax + X(x,y), \qquad \dot{y} = Y(x,y), \tag{3.1}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A is an $m \times m$-matrix, and $X(x, y)$, $Y(x, y)$ represent higher order nonlinear terms. If all eigenvalues of the matrix A have negative real parts, and $X(x, y)$, $Y(x, y)$ vanish when $x = 0$, then the solution $x = 0$, $y = 0$ of this system is stable with respect to (x, y), and asymptotically stable with respect to x. If a solution $(x(t), y(t))$ is close enough to the solution $x = 0$, $y = 0$, then

$$\lim_{t \to \infty} x(t) = 0, \qquad \lim_{t \to \infty} y(t) = c.$$

This theorem was used by a number of authors in analyzing stability of nonholonomic systems. See Karapetyan [1981], Markeev [1992], Zenkov, Bloch and Marsden [1998] and references therein. In particular, we stress that the conditions $X(0, y) = 0$ and $Y(0, y) = 0$ are valid for all systems considered in Bloch, Krishnaprasad, Marsden and Murray [1996] and in Zenkov, Bloch and Marsden [1998].

We begin with a description of substitutions that transform equations (2.4)–(2.7) into equations of the form (3.1). Consider an upright steady state motion of the unicycle represented by the relative equilibrium

$$\theta = 0, \qquad \alpha = 0, \qquad p_1 = 0, \qquad p_2 = p_2^0. \tag{3.2}$$

Put

$$p_1 = z_1 + I_0^{22} p_2^0 [\mathcal{A}\theta + \mathcal{B}(\alpha - \theta)], \qquad p_2 = p_2^0 + z_2.$$

Here and below all tensors and partial derivatives are evaluated at relative equilibrium (3.2). Equations (2.4)–(2.7) become

$$\begin{aligned}
g_{11}^0 \ddot{\theta} + g_{12}^0 \ddot{\alpha} = &- \frac{\partial I^{12}}{\partial \theta} p_2^0 \left[z_1 + I_0^{22} p_2^0 (\mathcal{A}\theta + \mathcal{B}(\alpha - \theta)) \right] \\
&- \frac{p_0^2}{2} \left[\frac{\partial^2 I^{22}}{\partial \theta^2} \theta + \frac{\partial^2 I^{22}}{\partial \theta \partial \alpha} \alpha \right] \\
&- \frac{\partial^2 U}{\partial \theta^2} \theta - \frac{\partial^2 U}{\partial \theta \partial \alpha} \alpha + \{\text{nonlinear terms}\}, && (3.3) \\
g_{12}^0 \ddot{\theta} + g_{22}^0 \ddot{\alpha} = &- \frac{\partial I^{12}}{\partial \alpha} p_2^0 \left[z_1 + I_0^{22} p_2^0 (\mathcal{A}\theta + \mathcal{B}(\alpha - \theta)) \right] \\
&- \frac{p_0^2}{2} \left[\frac{\partial^2 I^{22}}{\partial \alpha \partial \theta} \theta + \frac{\partial^2 I^{22}}{\partial \alpha^2} \alpha \right] \\
&- \frac{\partial^2 U}{\partial \alpha \partial \theta} \theta - \frac{\partial^2 U}{\partial \alpha^2} \alpha + u + \{\text{nonlinear terms}\}, && (3.4) \\
\dot{z}_1 = &\; Z_1(\theta, \alpha, \dot{\theta}, \dot{\alpha}, z), && (3.5) \\
\dot{z}_2 = &\; Z_2(\theta, \alpha, \dot{\theta}, \dot{\alpha}, z), && (3.6)
\end{aligned}$$

where

$$u = k_1\theta + k_2\alpha + k_3\dot{\theta} + k_4\dot{\alpha}$$

is *linear feedback control*. (The nonlinear terms in the α-equation may also contain nonlinear control inputs, but this is not assumed here—see also the remarks below.) Note that Z_1 and Z_2 vanish when $\dot{\theta} = \dot{\alpha} = 0$.

The shape equations (the first two of equations (2.4)–(2.7)) after being solved for $\ddot{\theta}$ and $\ddot{\alpha}$ become

$$\dot{v} = y, \qquad \dot{y} = Av + By + Cz + W(v, y, z),$$

where $v = (\theta, \alpha)$, $y = (\dot{\theta}, \dot{\alpha})$ and $W(v, y, z)$ represent the nonlinear terms. Suppose that we can choose a linear control which forces all eigenvalues of the matrix

$$\begin{pmatrix} 0 & I \\ A & B \end{pmatrix} \tag{3.7}$$

to belong to the left half plane (see theorem 4.1 below). Then $\det A \neq 0$, and we can find the solution $v = \chi(z)$ of the equation

$$Av + Cz + W(v, 0, z) = 0. \tag{3.8}$$

Introduce new variables x by $v = x + \chi(z)$. Then in the variables (x, y, z) equations (2.4)–(2.7) become

$$\begin{aligned} \dot{x} &= y + X(x, y, z), \\ \dot{y} &= Ax + By + Y(x, y, z), \\ \dot{z} &= Z(x, y, z), \end{aligned}$$

where

$$\begin{aligned} X(x, y, z) &= -\frac{\partial \chi}{\partial z} Z(x, y, z), \\ Y(x, y, z) &= A\chi(z) + Cz + W(x + \chi(z), y, z), \\ Z(x, y, z) &= Z(x + \chi(z), y, z). \end{aligned}$$

Observe that the nonlinear terms $X(x, y, z)$, $Y(x, y, z)$, and $Z(x, y, z)$ vanish when $x = 0$ and $y = 0$, because $Z(x, 0, z) \equiv 0$ and $A\chi(z) + Cz + W(\chi(z), 0, z) \equiv 0$. By the Lyapunov-Malkin theorem, equilibrium (3.2) is stable.

4 Existence of control

In this section we show how to choose the linear feedback control that forces the spectrum of matrix (3.7) to belong to the left half plane.

We note first that the coefficients of the characteristic polynomial $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$ of matrix (3.7) are linear functions in the gain parameters (k_1, k_2, k_3, k_4): There exist a matrix L and a vector M such that

$$a = Lk + M.$$

In the above formula $a = (a_1, a_2, a_3, a_4$ and $k = k_1, k_2, k_3, k_4$. Direct computation shows that $\det L$ is a rational function of the parameters of the system and p_2^0 and thus generically $\det L \neq 0$. The explicit formula however is very complex and we omit it here. Therefore the Routh-Hurwitz conditions

$$a_1 > 0, \qquad a_1a_2 - a_3 > 0, \qquad (a_1a_2 - a_3)a_3 - (a_1)^2a_4 > 0, \qquad a_4 > 0$$

for the spectrum of matrix (3.7) to be in the left half plane can always be satisfied by an appropriate choice of the gain parameters.

Summarizing, we have:

Theorem 4.1 *There exists a non-empty stability region $\mathcal{S}$ in the space of the gain parameters. For any $(k_1, k_2, k_3, k_4) \in \mathcal{S}$ the spectrum of (3.7) belongs to the left half plane and therefore by the Lyapunov-Malkin theorem the steady state motion (3.2) is stable.*

Remark. We emphasize that the Lyapunov-Malkin theorem can be used for nonlinear feedback stabilization. It extends a spectral stability condition to a nonlinear setting. We expect that the domain of the local coordinates (r, s) can be expanded by an appropriate choice of nonlinear control terms. The basin of attraction therefore may be enlarged. We intend to address this issue in a future publication.

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